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# Final Exam
# Mechatronics (NAMO05E)
# Friday, July 3, 2009 (9:00-12:00)

*Please write completely your name, student ID number and date of birth on this first page. For all subsequent pages, you only need to write your name.*

*You could choose to answer FIVE questions out of SIX available questions. If you decide to answer all six questions, the exam mark will be based only on the best five answered questions.*

*If you do not know the answer for one problem within a question, you could directly answer the subsequent problem in the same question. Do not waste your time. The questions are designed carefully to ensure that you can answer any problem within a question independently (except for questions 5 and 6).*

*If you are unclear about a specific problem, you could make your own assumptions and describe your assumptions in the beginning of your answer.*

*If there is not enough space for your answer, you could use blank paper provided at the end of this exam papers (put your name and the question number).*

*This is a CLOSED book exam.*

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## Question 1. (Total mark: 20) (Mechatronics systems)

Figure 1(a) shows the NASA Spirit Mars explorer which carries several scientific instruments on board. One of these instruments is the rock ablation tools (Figure 1(b) ). In order to collect the rock specimen properly, the controller on board must be able to sample the rock autonomously with minimal intervention from the earth. With the reference to Figure 2, we wish to identify variables that can be used in the positioning mechanisms of the rock ablation tools.

(a) (b)

*Figure 1. a). Spirit, the NASA Mars robotic rover. b). Rock ablation tools in the Spirit*

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a) Identify at least two possible measured variables (5 marks)

Answer 1a).
- Position of the rock (the target) relative to the rover
- Position (displacement or angular displacement) of every link in the robotic arms
- Velocity of every link in the arms
- Contact force with the rock

b) Identify at least two possible manipulated variables (5 marks)

Answer 1b).
- The electrical voltage of the electrical motor mechanism in the robotic arms
- Valve position of the hydraulic mechanism in the robotic arms
- The electrical voltage of the electrical motors in the rover wheels
- The electrical voltage of electrical motor that drives the vision systems, for stereoscopic measurement (for instance, zooming and rotating the camera)
- Valve position of the pneumatic or hydraulic braking systems of the rover

c) What are the possible choices of sensors and actuators that are related to the variables identified in a) and b)? (5 marks)

Answer 1c).
Possible sensors:
- For positional measurement: optical encoder, hall sensors, inductive sensor, camera vision systems, laser measurement systems
- For velocity: tachogenerator, hall sensors, inductive sensor
- For contact force: strain gauge, piezoresistive sensor or other pressure sensor devices

Possible actuators:
- Electrical motor
- Electrohydraulic servoactuator
- Pneumatic servoactuator

d) Identify at least one possible reference variable (5 marks)

Answer 1d).
Possible reference variable:
- The position and orientation of the desired location of the rock to be sampled
- The depth of the ablation (or cutting out)
- Location of the rock-of-interest
- Desired position and orientation of the rover in order to get a good working space for the robotic arms

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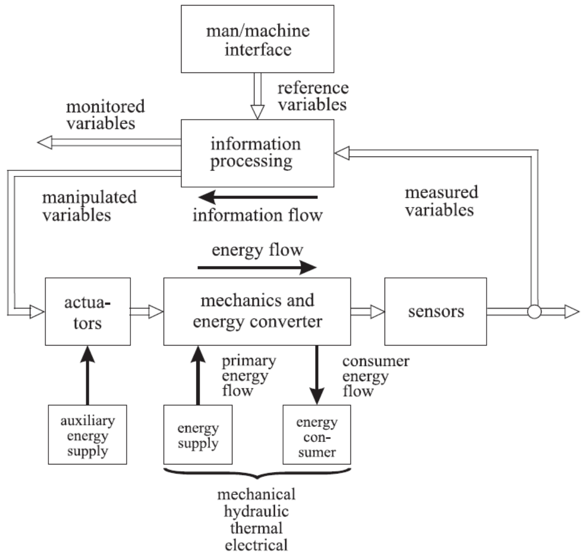



Figure 2. Mechatronics block diagram

# Question 2. (Total mark: 20) (Integrated systems modeling)

Consider the problem of controlling the position of ink cartridge in a printer as depicted in Figure 3. The cartridge is mounted in a conveyor belt that is driven by the DC motor.

Assume that all components are linear, i.e., the inductance is given by $L$ and the resistance is given by $R$. Derive the state equations of the full systems with the voltage $V$ as the input. Suppose that the output (or measured variable) is the position of the cartridge.

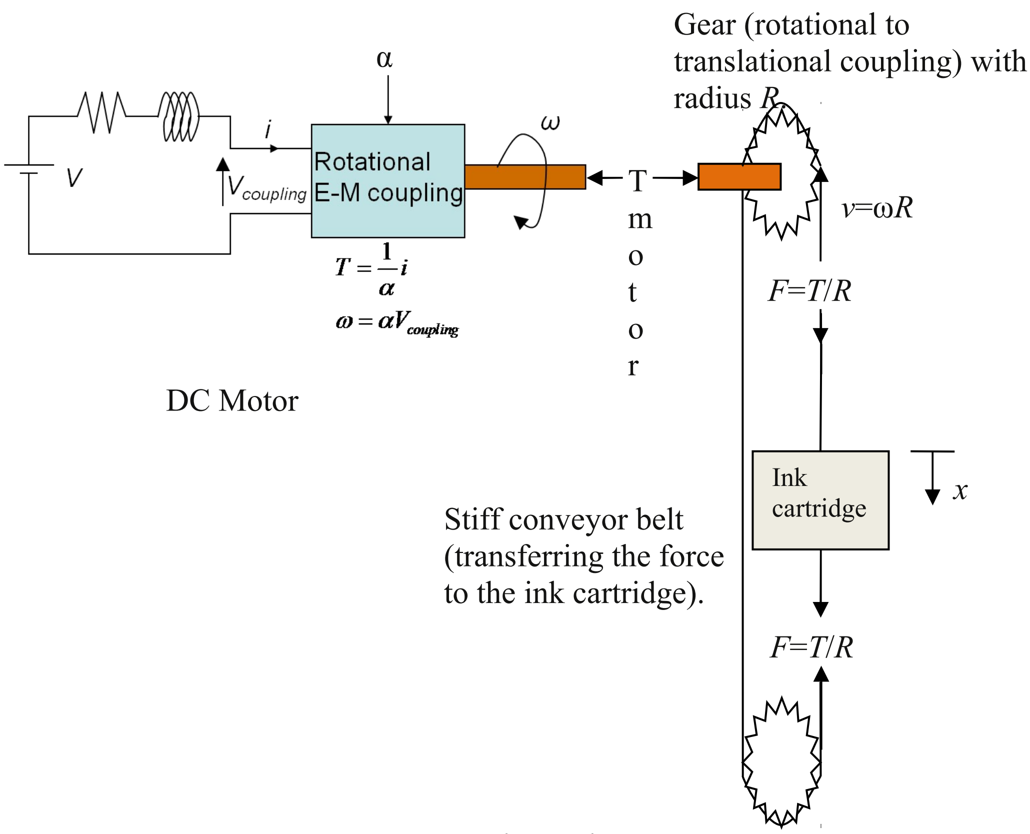



*Figure 3. Printer's ink cartridge positioning system.*

Answer 2).
In the question, it is not assumed that the rotating rod has moment of inertia. In this answer sheet, we will assume that the rod has an inertia of $I_{rod}$. However, if you answer this question without this assumption, it is still valid for this exam.

First, let us describe the differential equation of the DC motor. By Kirchoff's voltage law, we have:

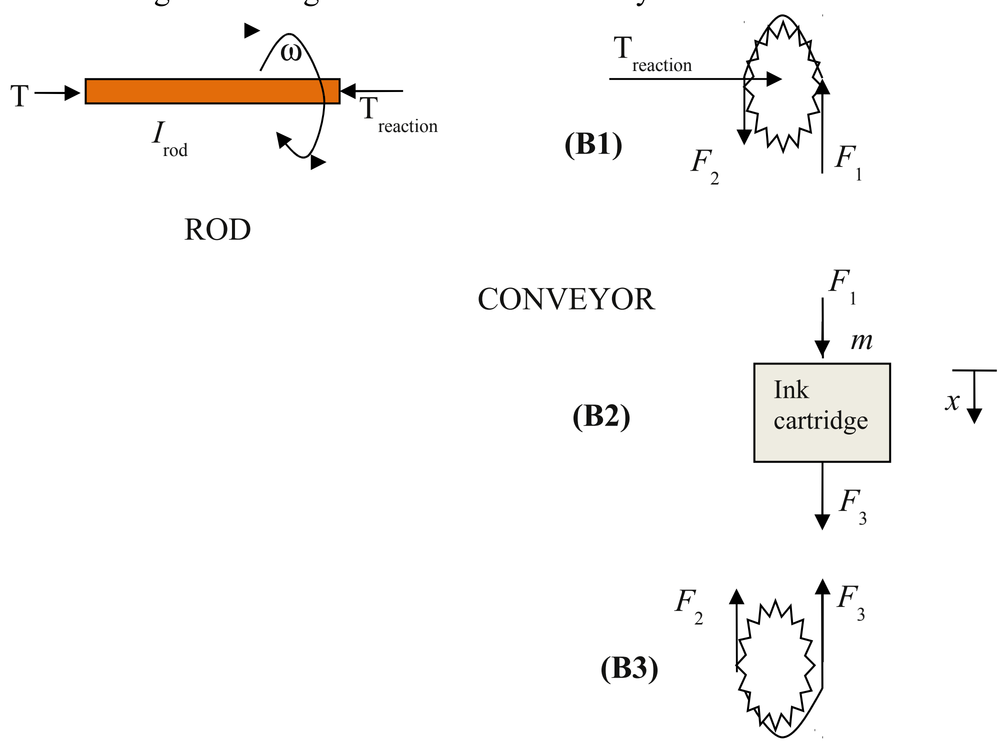
$$V = Ri + L\frac{di}{dt} + V_{coupling}$$

Substituting the relation of $V_{coupling}$ with ω in the rotational E-M coupling block, we get

$$V = Ri + L\frac{di}{dt} + \frac{1}{\alpha}\omega \qquad \textbf{(A1)}$$

Now, let us describe the mechanical equation of motions due to the gear, the cartridge and the rotating rod.

We redraw again the diagram for the mechanical systems:




The rotational motion of the rod is given by:

$$I_{rod}\frac{d\omega}{dt} = T - T_{reaction} = \frac{1}{\alpha}i - T_{reaction} \qquad \textbf{(A2)}$$

where the last equation is due to the relation between $T$ and the current $i$ in the rotation E-M coupling block.

From the element **(B1)** (i.e., the upper wheel) we have that (according to Newton's law action and reaction law): $0 = T_{reaction} - F_1 R - F_2 R$
From the element **(B3)** (i.e., the lower wheel) we also have $0 = F_2 - F_3$
And for the element **(B2)** (i.e., the cartridge) the Newton's law yields

$$m\frac{d^2x}{dt^2} = F_3 + F_1$$

Combining the equations from the elements (B1), (B2) and (B3), we obtain:

$$T_{reaction} = F_1R + F_2R = F_1R + F_3R = mR\frac{d^2x}{dt^2} \qquad \textbf{(A3)}$$

Substituting **(A3)** back to **(A2)**, in order to remove $T_{\text{reaction}}$, we obtain

$$\begin{aligned} & I_{rod}\frac{d\omega}{dt} = \frac{1}{\alpha}i - mR\frac{d^2x}{dt^2} \\ \Leftrightarrow\ & I_{rod}\frac{d\omega}{dt} + mR\frac{d^2x}{dt^2} = \frac{1}{\alpha}i \end{aligned} \qquad \textbf{(A4)}$$

Now, there are two possibilities for the choice of the generalized variables for the mechanical systems. One uses the angular displacement of the rod $\theta$, the other uses the linear displacement of the cartridge $x$. Both of the are related by $\frac{dx}{dt} = R\frac{d\theta}{dt}$ and $x = R\theta$.

Let us use the cartridge position $x$ as the generalized coordinate. In this case, **(A4)** becomes

$$I_{rod}\frac{1}{R}\frac{d^2x}{dt^2} + mR\frac{d^2x}{dt^2} = \frac{1}{\alpha}i$$

$$\Leftrightarrow \left(\frac{I_{rod}}{R^2} + m\right)\frac{d^2x}{dt^2} = \frac{1}{\alpha R}i$$

$$\Leftrightarrow \frac{d^2x}{dt^2} = \left(\frac{I_{rod}}{R^2} + m\right)^{-1}\frac{1}{\alpha R}i$$

and **(A1)** becomes

$$V = Ri + L\frac{di}{dt} + \frac{1}{\alpha R}\frac{dx}{dt}$$

The above two equations constitutes the motion of equations of the ink cartridge system. It can also be written in the state space form. For example, if we choose the state: $x_1 = i,\ x_2 = x,\ x_3 = \frac{dx}{dt}$. In this case, the above equations become:

$$V = Ri + L\frac{di}{dt} + \frac{1}{\alpha R}\frac{dx}{dt} \quad \Leftrightarrow \quad \dot{x}_1 = -\frac{R}{L}x_1 - \frac{1}{\alpha RL}x_3 + \frac{1}{L}V \quad \text{and}$$

$$\frac{d^2x}{dt^2} = \left(\frac{I_{rod}}{R^2} + m\right)^{-1}\frac{1}{\alpha R}i \quad \Leftrightarrow \quad \dot{x}_3 = \left(\frac{I_{rod}}{R^2} + m\right)^{-1}\frac{1}{\alpha R}x_1.$$

These equations, together with the fact that $\dot{x}_2 = x_3$, we have the following state equations for the ink cartridge systems:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 & -\frac{1}{\alpha RL} \\ 0 & 0 & 1 \\ \left(\frac{I_{rod}}{R^2}+m\right)^{-1}\frac{1}{\alpha R} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} V$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note that if you do not assume the moment of inertia of the rod, then you would get the above state equations with $I_{rod} = 0$.

## Question 3. (Total mark: 20) (General Questions)

a) For state equations with nonlinear terms, describe at least two situations where linearization technique is not applicable. (5 marks)

Answer 3a).

- Time varying systems
- Systems with hysteretic element
- If it were used for designing controller to track non-constant reference signal (to track time-varying signal)
- If it contains multi-valued functions
- If there is discontinuous function in the system's equations.

b) What can be the input and output variables in a motorized prosthetic arm? (5 marks)

Answer 3b).
The possible input variable: The voltage in the electrical motor or in the electrohydraulic servoactuator.

The possible output variables: The position and the velocity of the link in the arm, the current in the motor.

c) What are the T-type and A-type variables in the mechanical and electrical systems? (5 marks)

Answer 3c).
The T-type variables: In mechanical systems is the Force, in electrical systems is the current.
The A-type variables: In mechanical systems is the velocity, in the electrical systems is the voltage.

d) Give an example of T-type and A-type element in the mechanical systems. (5 marks)

Answer 3d).
The example of T-type element: the spring
The example of A-type element: the mass

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## Question 4. (Total mark: 20) (Nonlinear systems and linearization)

Consider the nonlinear state equations of the pendulum (shown in Figure 4), $\dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{g}{l}\sin x_1 + \frac{1}{ml^2}T$

where $x_1$ denotes the angle, $x_2$ denotes the angular velocity, $g$ is the gravity acceleration, $m$ is the mass of the ball, $l$ is the length of the string and $T$ is the control torque. Suppose that we can only measure the angle $x_1$.

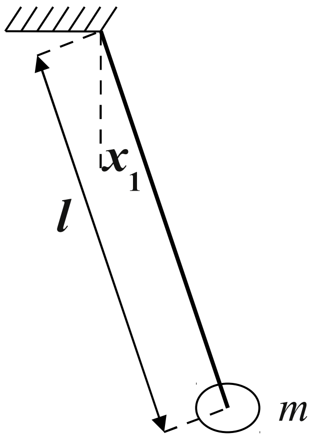


Figure 4. The pendulum

a) Show that the linearization of the system around the equilibrium point $\left(x_1^* = \frac{2\pi}{3}, x_2^* = 0, T^* = mgl\frac{1}{2}\sqrt{3}\right)$ is given by $\dot{\tilde{x}}_1 = \tilde{x}_2$, $\dot{\tilde{x}}_2 = \frac{g}{2l}\tilde{x}_1 + \frac{1}{ml^2}\tilde{T}$ and $e = \tilde{x}_1$, where $\tilde{x}_1 = x_1 - x_1^*$, $\tilde{x}_2 = x_2 - x_2^*$, $\tilde{T} = T - T^*$ and $e$ is the output of the linearized system (the error signal). (Note that the above linearized system is also called error system.) (5 marks)

Answer 4a).
We start directly from the Step 3, in the linearization process discussed in the lecture slide Week 2 Day 2.
Using $\tilde{x}_1 = x_1 - x_1^*, \tilde{x}_2 = x_2 - x_2^*, \tilde{T} = T - T^*$ and using the original state equations, it follows that

$\dot{\tilde{x}}_1 = \dot{x}_1 - \dot{x}_1^*,$

$\dot{\tilde{x}}_2 = \dot{x}_2 - \dot{x}_2^*$

Using the fact that $\dot{x}_1^* = 0, \dot{x}_2^* = 0, \dot{x}_1 = x_2, \dot{x}_2 = -\frac{g}{l}\sin x_1 + \frac{1}{ml^2}T$, the above equations become

$$\dot{\tilde{x}}_1 = x_2 = \tilde{x}_2 + x_2^* = \tilde{x}_2$$

$$\dot{\tilde{x}}_2 = -\frac{g}{l}\sin x_1 + \frac{1}{ml^2}T = -\frac{g}{l}\sin(\tilde{x}_1 + x_1^*) + \frac{1}{ml^2}(\tilde{T} + T^*)$$

Let us linearize the above equations around $\tilde{x}_1 = 0, \tilde{x}_2 = 0, \tilde{T} = 0$ as follows (by the Taylor expansions of the above right hand side equations around these points):

$$\dot{\tilde{x}}_1 = \tilde{x}_2$$

$$\dot{\tilde{x}}_2 = -\frac{g}{l}\sin(x_1^*) - \left( \frac{d}{d\tilde{x}_1}\left[\frac{g}{l}\sin(\tilde{x}_1 + x_1^*)\right]\Bigg|_{\substack{\tilde{x}_1=0,\\ \tilde{x}_2=0,\\ \tilde{T}=0}} \right)\tilde{x}_1 + \frac{1}{ml^2}(\tilde{T} + T^*)$$

$$= -\frac{g}{l}\sin\left(\frac{2\pi}{3}\right) - \frac{g}{l}\cos\left(\frac{2\pi}{3}\right)\tilde{x}_1 + \frac{1}{ml^2}\left(\tilde{T} + mgl\frac{1}{2}\sqrt{3}\right)$$

$$= -\frac{g}{l}\frac{1}{2}\sqrt{3} + \frac{g}{l}\frac{1}{2}\tilde{x}_1 + \frac{1}{ml^2}\tilde{T} + \frac{g}{l}\frac{1}{2}\sqrt{3}$$

$$= \frac{g}{2l}\tilde{x}_1 + \frac{1}{ml^2}\tilde{T}$$

The output equation is just the error signal, which is

$$e = x_1 - x_1^* = \tilde{x}_1$$

b) Show that the transfer function of the state equations above (from the incremental torque $\tilde{T}$ to the error signal $e$) is given by:

$$\frac{E(s)}{\tilde{T}(s)} = \frac{1/ml^2}{s^2 - g/2l}$$

where $E(s)$ and $\tilde{T}(s)$ are the Laplace transform of the error and the incremental torque signals. (5 marks.)

Answer 4b).

There are several ways to show this. The one that will be described here is based on the state equations. The other one can be obtained by directly applying Laplace transformation to the input-output equations of the error system (hence, you first have to find the input-output differential equations of the system).

From the linearize state equations above, it can be written compactly in the following form:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{2l} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \tilde{T}$$

$$e = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

Hence, we have $A = \begin{bmatrix} 0 & 1 \\ \frac{g}{2l} & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = 0$ in the standard state-space equation form. By applying the formula for transfer function using the above matrices *A,B,C,D*, then

$$\frac{E(s)}{\tilde{T}(s)} = C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ -\frac{g}{2l} & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

$$= \frac{1}{s^2 - \frac{g}{2l}} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ \frac{g}{2l} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

$$= \frac{1/ml^2}{s^2 - \frac{g}{2l}}$$

c) Based on the equations in a) or in b), can the error system be stabilized by proportional feedback only? Give your reasons. (5 marks) (Note that you do not have to answer a) and b) in order to answer this question!!! You can simply use the state equation or the transfer function provided in a) or b).)

Answer 4c).

From the transfer function of the error systems, it can be checked that the characteristic polynomial of the closed-loop system with the proportional feedback is given by

$$\chi(s) = s^2 - \frac{g}{2l} + \frac{K_p}{ml^2}.$$

(The computation of the above polynomial follows the same way as in the exercises, where we use only proportional feedback $K_p$).

By checking the coefficient in the characteristics polynomial, the coefficient that corresponds to $s^1$ is equal to 0. Hence, according to the Routh-Hurwitz criterion, it is not stable. This means that the error system cannot be stabilized by using only proportional feedback. Note that the closed-loop system can only be made marginally stable with one or more poles located at the imaginary axis.

d) Can the error equation be stabilized by proportional+derivative feedback (i.e., we can have full-state feedback or PD controller) (5 marks)

Answer 4d).

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Similarly, from the transfer function of the error systems, it can be checked that the characteristic polynomial of the closed-loop system with the proportional feedback is given by

$$\chi(s) = s^2 + \frac{K_d}{ml^2}s - \frac{g}{2l} + \frac{K_p}{ml^2}.$$

Now, by assigning $K_d>0$ and $K_p>mgl/2$, all the coefficients of the above polynomial are positive, and by checking the Routh-Hurwitz matrix, the same condition ensure that components in the first column of Routh-Hurwitz array are positive. Therefore, it can be stabilized by the proportional+derivative feedback.

# Question 5. (Total mark: 20)

In the control of space satellite, the orientation of the satellite can be adjusted by controlling the angles of the flying wheels. A simplified model of the mechanism is shown in Figure 5. The first component is the leftmost bar with mass, length and inertia be $m_1$, $l_1$ and $I_1$ respectively. The second component is the rightmost bar with mass, length and inertia be $m_2$, $l_2$ and $I_2$ respectively. Both components has their centre of the mass at its centre.

Suppose that the system is placed on a FRICTIONLESS table where it can move to any position in $x$ and $y$ direction freely, without any friction force. The control torque is provided by the motor attached to the joint. The direction of control torque $T$ is towards the table, i.e., positive torque ($T$>0) induces clockwise rotation to the first component (if the second component is fixed to the table). The motor is NOT fixed/attached to the table. By applying torque to the joint, both components (the wheel and the bar) can rotate and move in $x$ and $y$ direction. The gravity force does not affect the system.

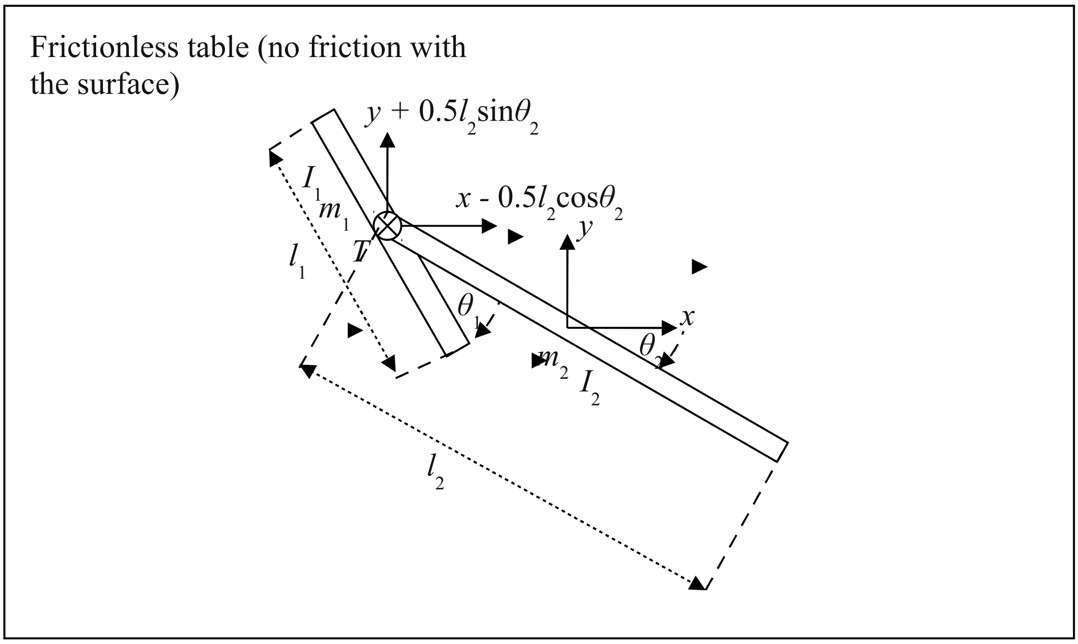


Figure 5. A simplified model of space craft orientation control using flywheel.

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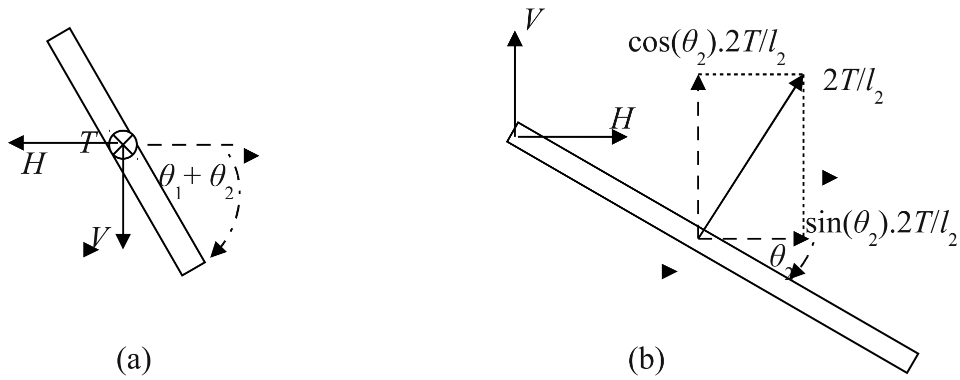


Figure 6. (a) The first bar (the leftmost component in Figure 5) with the reaction forces *V* and *H* and the control torque *T*; (b) The second bar (the rightmost component in Figure 5) with the reaction forces *V* and *H,* and the reaction force $2T/l_2$ due to the control torque in the joint.

We wish to derive the equations of motion for the system as shown in Figure 5 using the Newtonian methods with the generalized coordinates $x$, $y$, $\theta_1$, $\theta_2$. The forces and torques exerted to each component are shown in Figure 6.

a) Using the Cartesian coordinate of the first (leftmost) bar ($x - 0.5l_2 \cos \theta_2$, $y + 0.5l_2 \sin \theta_2$), compute the horizontal and the vertical Newton's laws. (4 marks)

Answer 5a).

The horizontal and the vertical Newton's law for the first bar are:

$$m_1\ddot{x}_1 = -H$$
$$\Leftrightarrow m_1 \frac{d^2}{dt^2}\left(x - 0.5l_2 \cos\theta_2\right) = -H$$
$$\Leftrightarrow m_1\left(\ddot{x} + 0.5l_2 \sin(\theta_2)\ddot{\theta}_2 + 0.5l_2 \cos(\theta_2)\dot{\theta}_2^2\right) = -H$$
$$m_1\ddot{y}_1 = -V$$
$$\Leftrightarrow m_1 \frac{d^2}{dt^2}\left(y + 0.5l_2 \sin\theta_2\right) = -V$$
$$\Leftrightarrow m_1\left(\ddot{y} + 0.5l_2 \cos(\theta_2)\ddot{\theta}_2 - 0.5l_2 \sin(\theta_2)\dot{\theta}_2^2\right) = -V$$

b) Using the Cartesian coordinate of the second (rightmost) bar ($x$, $y$), compute the horizontal and the vertical Newton's laws. (4 marks)

Answer 5b).
The horizontal and vertical Newton's laws for the second bar are:

$$m_2\ddot{x}_2 = \frac{2T}{l_2}\sin\theta_2 + H$$
$$\Leftrightarrow m_2\ddot{x} = \frac{2T}{l_2}\sin\theta_2 + H$$
$$m_2\ddot{y}_2 = \frac{2T}{l_2}\cos\theta_2 + V$$
$$\Leftrightarrow m_2\ddot{y} = \frac{2T}{l_2}\cos\theta_2 + V$$

c) Compute the rotational Newton's laws of each component at its centre of mass. (4 marks) (Note that the equation for the first bar is the first equation (out of four equations) of motion of the system.)

Answer 5c).

The equation of rotational Newton's law for the first bar is

$$I_1\left(\ddot{\theta}_1+\ddot{\theta}_2\right)=T$$

since the total rotation of the with respect to the reference frame xy is $(\theta_1+\theta_2)$.

The equation of the rotational Newton's law for the second bar is

$$I_2\ddot{\theta}_2=V\frac{l_2}{2}\cos\theta_2+H\frac{l_2}{2}\sin\theta_2 \qquad \textbf{(C1)}$$

d) Using the answers in a) and b), compute the second and the third equations of motion by eliminating the reaction forces $V$ and $H$. (4 marks)

Answer 5d).

Substituting the horizontal Newton's equations of the first bar to the second bar, we get

$$m_2\ddot{x}+m_1\left(\ddot{x}+0.5l_2\sin(\theta_2)\ddot{\theta}_2+0.5l_2\cos(\theta_2)\dot{\theta}_2^2\right)=\frac{2T}{l_2}\sin\theta_2$$

And substituting the vertical Newton's equations of the first bar to the second bar, we get

$$m_2\ddot{y}+m_1\left(\ddot{y}+0.5l_2\cos(\theta_2)\ddot{\theta}_2-0.5l_2\sin(\theta_2)\dot{\theta}_2^2\right)=\frac{2T}{l_2}\cos\theta_2$$

e) Using the answer in a) and the rotational Newton's law for the second bar, compute the last equation of motion of the system. (4 marks.)

Answer 5e).

Substituting both the horizontal and the vertical Newton's law of the first bar, to the equation **(C1)**, we get

$$\begin{aligned}I_2\ddot{\theta}_2 &= -m_1\left(\ddot{y}+0.5l_2\cos(\theta_2)\ddot{\theta}_2-0.5l_2\sin(\theta_2)\dot{\theta}_2^2\right)\frac{l_2}{2}\cos\theta_2\\ &\quad -m_1\left(\ddot{x}+0.5l_2\sin(\theta_2)\ddot{\theta}_2+0.5l_2\cos(\theta_2)\dot{\theta}_2^2\right)\frac{l_2}{2}\sin\theta_2\\ \Leftrightarrow I_2\ddot{\theta}_2 &= -m_1\frac{l_2}{2}\left(\ddot{y}\cos\theta_2+0.5l_2\cos^2(\theta_2)\ddot{\theta}_2-0.5l_2\cos(\theta_2)\sin(\theta_2)\dot{\theta}_2^2\right)\\ &\quad -m_1\frac{l_2}{2}\left(\ddot{x}\sin\theta_2+0.5l_2\sin^2(\theta_2)\ddot{\theta}_2+0.5l_2\cos(\theta_2)\sin(\theta_2)\dot{\theta}_2^2\right)\\ \Leftrightarrow I_2\ddot{\theta}_2 &= -m_1\frac{l_2}{2}\left(\ddot{y}\cos\theta_2+\ddot{x}\sin\theta_2+0.5l_2\ddot{\theta}_2\right)\end{aligned}$$

Or, equivalently:

$$I_2\ddot{\theta}_2+m_1\frac{l_2}{2}\left(\ddot{y}\cos\theta_2+\ddot{x}\sin\theta_2+0.5l_2\ddot{\theta}_2\right)=0$$

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In the last question, where we are going to use the Euler-Lagrange equation, the second equation of motion there (in the answer of Question 6) is equal to the above equation + the rotational equation of the first bar (from the answer of Question 5c).

More precisely, let us add the above equation with the rotational equation of the first bar as follows:

$$I_2\ddot{\theta}_2 + m_1\frac{l_2}{2}\left(\ddot{y}\cos\theta_2 + \ddot{x}\sin\theta_2 + 0.5l_2\ddot{\theta}_2\right) + I_1\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) = T$$

It will be shown later (in the next question) that the above equation is the same as the equation of motion obtained from Euler Lagrange method for the generalized coordinate $\theta_2$.

# Question 6. (Total mark: 20)

Consider the same system as detailed in Question 5.

Derive the equations of motion using the Euler-Lagrange methods with the generalized coordinates $x$, $y$, $\theta_1$, $\theta_2$. (Hint: compute the kinetic energy and derive the equations of motion via Euler-Lagrange equation with generalized external forces/torques).

Answer 6).

Let us define the kinetic energy of the system:

$$E_k = \frac{1}{2}m_1\left(\frac{dx_1}{dt}\right)^2 + \frac{1}{2}m_1\left(\frac{dy_1}{dt}\right)^2 + \frac{1}{2}m_2\left(\frac{dx_2}{dt}\right)^2 + \frac{1}{2}m_2\left(\frac{dy_2}{dt}\right)^2 + \frac{1}{2}I_1\left(\frac{d(\theta_1+\theta_2)}{dt}\right)^2 + \frac{1}{2}I_2$$

Substituting $x_1 = x - \frac{l_2}{2}\cos\theta_2$, $y_1 = y + \frac{l_2}{2}\sin\theta_2$, $x_2 = x$ and $y_2 = y$ to the above equation, we obtain:

$$E_k = \frac{1}{2}m_1\left(\dot{x} + \frac{l_2}{2}\sin(\theta_2)\dot{\theta}_2\right)^2 + \frac{1}{2}m_1\left(\dot{y} + \frac{l_2}{2}\cos(\theta_2)\dot{\theta}_2\right)^2$$
$$+ \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}I_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \frac{1}{2}I_2\dot{\theta}_2{}^2$$

The generalized coordinate is $q = \begin{bmatrix}\theta_1\\ \theta_2\\ x\\ y\end{bmatrix}$ and the (global) spatial coordinate of components (describing the positions and orientations of both the first and the second bar) with respect to the reference frame can be given by

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$$r = \begin{bmatrix} \theta_1 + \theta_2 \\ \theta_2 \\ x \\ y \\ x - \frac{l_2}{2}\cos\theta_2 \\ y + \frac{l_2}{2}\sin\theta_2 \end{bmatrix} \text{ with the corresponding external force/torque } F_{ext} = \begin{bmatrix} T \\ 0 \\ \frac{2T}{l_2}\sin\theta_2 \\ \frac{2T}{l_2}\cos\theta_2 \\ 0 \\ 0 \end{bmatrix}.$$

(It can be observed that the external force/torque are those that acts on the centre of the mass of each bar, and they are closely related to the global spatial coordinate).

Now, let us describe the equation of motion using Euler-Lagrange for the first generalized coordinate $\theta_1$. We precede this by computing each component in the Euler-Lagrange equation:

$$\frac{\partial E_k}{\partial \dot{\theta}_1} = I_1(\dot{\theta}_1 + \dot{\theta}_2),$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\theta}_1}\right) = I_1(\ddot{\theta}_1 + \ddot{\theta}_2),$$

$$\frac{\partial E_k}{\partial \theta_1} = 0$$

$$F_{ext}^T \frac{\partial r}{\partial \theta_1} = \begin{bmatrix} T & 0 & \frac{2T}{l_2}\sin\theta_2 & \frac{2T}{l_2}\cos\theta_2 & 0 & 0 \end{bmatrix} \frac{d}{d\theta_1}\left(\begin{bmatrix} \theta_1 + \theta_2 \\ \theta_2 \\ x \\ y \\ x - \frac{l_2}{2}\cos\theta_2 \\ y + \frac{l_2}{2}\sin\theta_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} T & 0 & \frac{2T}{l_2}\sin\theta_2 & \frac{2T}{l_2}\cos\theta_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= T$$

Based on the above equations, the Euler-Lagrange for the first generalized coordinate:

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\theta}_1}\right) - \frac{\partial E_k}{\partial \theta_1} = F_{ext}^T \frac{\partial r}{\partial \theta_1}$$

$$\Leftrightarrow I_1(\ddot{\theta}_1 + \ddot{\theta}_2) = T$$

Now, let us compute the Euler-Lagrange for the second generalized coordinate $\theta_2$.

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Again, we compute first each component required in the Euler-Lagrange equation.

$$\frac{\partial E_k}{\partial \dot{\theta}_2} = I_1(\dot{\theta}_1 + \dot{\theta}_2) + I_2\dot{\theta}_2 + m_1\left(\dot{x} + \frac{l_2}{2}\sin(\theta_2)\dot{\theta}_2\right)\frac{l_2}{2}\sin(\theta_2)$$
$$+ m_1\left(\dot{y} + \frac{l_2}{2}\cos(\theta_2)\dot{\theta}_2\right)\frac{l_2}{2}\cos(\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\theta}_2}\right) = I_1(\ddot{\theta}_1 + \ddot{\theta}_2) + I_2\ddot{\theta}_2 + m_1\frac{l_2}{2}\left(\ddot{x} + \frac{l_2}{2}\cos(\theta_2)\dot{\theta}_2^2 + \frac{l_2}{2}\sin(\theta_2)\ddot{\theta}_2\right)\sin(\theta_2)$$
$$+ m_1\frac{l_2}{2}\left(\dot{x} + \frac{l_2}{2}\sin(\theta_2)\dot{\theta}_2\right)\cos(\theta_2)\dot{\theta}_2$$
$$+ m_1\frac{l_2}{2}\left(\ddot{y} - \frac{l_2}{2}\sin(\theta_2)\dot{\theta}_2^2 + \frac{l_2}{2}\cos(\theta_2)\ddot{\theta}_2\right)\cos(\theta_2)$$
$$- m_1\frac{l_2}{2}\left(\dot{y} + \frac{l_2}{2}\cos(\theta_2)\dot{\theta}_2\right)\sin(\theta_2)\dot{\theta}_2$$
$$= I_1(\ddot{\theta}_1 + \ddot{\theta}_2) + I_2\ddot{\theta}_2 + m_1\frac{l_2}{2}\sin(\theta_2)\ddot{x} + m_1\frac{l_2}{2}\cos(\theta_2)\ddot{y}$$
$$+ m_1\frac{l_2^2}{4}\ddot{\theta}_2 + m_1\frac{l_2}{2}\cos(\theta_2)\dot{x}\dot{\theta}_2 - m_1\frac{l_2}{2}\sin(\theta_2)\dot{y}\dot{\theta}_2$$

$$\frac{\partial E_k}{\partial \theta_2} = m_1\left(\dot{x} + \frac{l_2}{2}\sin(\theta_2)\dot{\theta}_2\right)\frac{l_2}{2}\cos(\theta_2)\dot{\theta}_2 - m_1\left(\dot{y} + \frac{l_2}{2}\cos(\theta_2)\dot{\theta}_2\right)\frac{l_2}{2}\sin(\theta_2)\dot{\theta}_2$$
$$= m_1\frac{l_2}{2}\cos(\theta_2)\dot{x}\dot{\theta}_2 - m_1\frac{l_2}{2}\sin(\theta_2)\dot{y}\dot{\theta}_2$$

$$F_{ext}^T\frac{\partial r}{\partial \theta_2} = \begin{bmatrix} T & 0 & \frac{2T}{l_2}\sin\theta_2 & \frac{2T}{l_2}\cos\theta_2 & 0 & 0 \end{bmatrix}\frac{d}{d\theta_2}\left(\begin{bmatrix} \theta_1 + \theta_2 \\ \theta_2 \\ x \\ y \\ x - \frac{l_2}{2}\cos\theta_2 \\ y + \frac{l_2}{2}\sin\theta_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} T & 0 & \frac{2T}{l_2}\sin\theta_2 & \frac{2T}{l_2}\cos\theta_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ \frac{l_2}{2}\sin\theta_2 \\ \frac{l_2}{2}\cos\theta_2 \end{bmatrix}$$

$$= T$$

Based on the above equations, the second equation of motion is

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$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{\theta}_2}\right)-\frac{\partial E_k}{\partial \theta_2}=F_{ext}^T\frac{\partial r}{\partial \theta_2}$$

$$\Leftrightarrow I_1\left(\ddot{\theta}_1+\ddot{\theta}_2\right)+I_2\ddot{\theta}_2+m_1\frac{l_2}{2}\sin(\theta_2)\ddot{x}+m_1\frac{l_2}{2}\cos(\theta_2)\ddot{y}+m_1\frac{l_2^2}{4}\ddot{\theta}_2=T$$

Now we will compute the Euler-Lagrange equation for the third generalized coordinate $x$. Again, we calculate each component of Euler-Lagrange equation:

$$\frac{\partial E_k}{\partial \dot{x}}=m_2\dot{x}+m_1\left(\dot{x}+\frac{l_2}{2}\sin(\theta_2)\dot{\theta}_2\right)$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{x}}\right)=m_2\ddot{x}+m_1\left(\ddot{x}+\frac{l_2}{2}\cos(\theta_2)\dot{\theta}_2^2+\frac{l_2}{2}\sin(\theta_2)\ddot{\theta}_2\right)$$

$$\frac{\partial E_k}{\partial x}=0$$

$$F_{ext}^T\frac{\partial r}{\partial x}=\begin{bmatrix}T & 0 & \frac{2T}{l_2}\sin\theta_2 & \frac{2T}{l_2}\cos\theta_2 & 0 & 0\end{bmatrix}\frac{d}{dx}\left(\begin{bmatrix}\theta_1+\theta_2\\ \theta_2\\ x\\ y\\ x-\frac{l_2}{2}\cos\theta_2\\ y+\frac{l_2}{2}\sin\theta_2\end{bmatrix}\right)$$

$$=\begin{bmatrix}T & 0 & \frac{2T}{l_2}\sin\theta_2 & \frac{2T}{l_2}\cos\theta_2 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\\0\\1\\0\end{bmatrix}$$

$$=\frac{2T}{l_2}\sin\theta_2$$

Based on the above equations, the third equation of motion is

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{x}}\right)-\frac{\partial E_k}{\partial x}=F_{ext}^T\frac{\partial r}{\partial x}$$

$$\Leftrightarrow m_2\ddot{x}+m_1\left(\ddot{x}+\frac{l_2}{2}\cos(\theta_2)\dot{\theta}_2^2+\frac{l_2}{2}\sin(\theta_2)\ddot{\theta}_2\right)=\frac{2T}{l_2}\sin\theta_2$$

It remains to find the last Euler-Lagrange equation for the fourth generalized coordinate $y$. Each component of the Euler Lagrange equations for this coordinate are:

$$\frac{\partial E_k}{\partial \dot{y}}=m_2\dot{y}+m_1\left(\dot{y}+\frac{l_2}{2}\cos(\theta_2)\dot{\theta}_2\right)$$

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{y}}\right)=m_2\ddot{y}+m_1\left(\ddot{y}-\frac{l_2}{2}\sin(\theta_2)\dot{\theta}_2^2+\frac{l_2}{2}\cos(\theta_2)\ddot{\theta}_2\right)$$

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$$\frac{\partial E_k}{\partial y} = 0$$

$$F_{ext}^T \frac{\partial r}{\partial y} = \begin{bmatrix} T & 0 & \frac{2T}{l_2}\sin\theta_2 & \frac{2T}{l_2}\cos\theta_2 & 0 & 0 \end{bmatrix} \frac{d}{dy}\left( \begin{bmatrix} \theta_1 + \theta_2 \\ \theta_2 \\ x \\ y \\ x - \frac{l_2}{2}\cos\theta_2 \\ y + \frac{l_2}{2}\sin\theta_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} T & 0 & \frac{2T}{l_2}\sin\theta_2 & \frac{2T}{l_2}\cos\theta_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{2T}{l_2}\cos\theta_2$$

Then, the final equation of motion is

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{y}}\right) - \frac{\partial E_k}{\partial y} = F_{ext}^T \frac{\partial r}{\partial y}$$

$$\Leftrightarrow m_2 \ddot{y} + m_1\left(\ddot{y} - \frac{l_2}{2}\sin(\theta_2)\dot{\theta}_2^2 + \frac{l_2}{2}\cos(\theta_2)\ddot{\theta}_2\right) = \frac{2T}{l_2}\cos\theta_2$$

**Additional blank paper:**